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Technical Note

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THE BUCKET AND THE BUNCH WITH A SECOND RF HARMONIC

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In the presence of a second RF harmonic the voltage acting on a particle is

(1) $$eV = eV_1 \sin(\omega_{RF}t + \varphi_1) + eV_2 \sin(2\omega_{RF}t + \varphi_2).$$

With the positions

(2) $$V_0 \equiv V_1, \quad \chi = \frac{V_2}{V_o}, \quad \phi = \omega_{RF}t + \varphi_1, \quad \Delta\varphi = \varphi_2 - 2\varphi_1,$$

the differential equation for the energy change during the acceleration becomes

(3) $$\tau \frac{d}{dt}(\Delta E) = eV_o\{\sin\phi - \sin\phi_s + \chi[\sin(2\phi + \Delta\varphi) - \sin(2\phi_s + \Delta\varphi)]\}$$

with ϕ_s the synchronous phase, to be calculated by solving the transcendental Equation

(4) $$\boxed{\frac{\Delta E_s}{eV_0} = \sin\phi_s + \chi\sin(2\phi_s + \Delta\varphi)}.$$

To find the bucket separatrix -the "bucket"-, let us start from the differential equation for the phase

(5) $$\frac{d^2\phi}{dt^2} = \frac{2\pi}{\tau^2}\frac{h\eta}{\beta^2}\frac{eV_0}{E_s}f_b,$$

with f_b the function on the rhs of Eq. (3)

(6) $$f_b = \sin\phi - \sin\phi_s + \chi[\sin(2\phi + \Delta\varphi) - \sin(2\phi_s + \Delta\varphi)],$$

and with the harmonic number h, the dispersion η, and the relativistic velocity β evaluated for the synchronous particle (we dropped the subscript "s").

An integral of motion is found by integration in the standard way[1]

(7) $$\int \frac{d^2\phi}{dt^2}\frac{d\phi}{dt}dt = \frac{2\pi}{\tau^2}\frac{h\eta}{\beta^2}\frac{eV_0}{E_s}\int f_b \frac{d\phi}{dt}dt.$$

The integration yields the following

(8) $$\frac{K}{2}\left(\frac{\Delta E}{E_s}\right)^2 + eV_0\tilde{b}(\phi,\phi_s) = H = const,$$

with

(9) $$K = 2\pi\frac{h\eta}{\beta^2}E_s,$$

and

(10) $$\tilde{b}(\phi,\phi_s) = \cos\phi + \phi\sin\phi_s + \chi\left[\tfrac{1}{2}\cos(2\phi+\Delta\varphi) + \phi\sin(2\phi_s+\Delta\varphi)\right].$$

The constant H of Eq.(3) is the Hamiltonian of the system

(11) $$H = \tfrac{1}{2}KW^2 + U,$$

and the bucket is found using the two fixed points ϕ_s and π - ϕ_s of the distribution[2]

(12) $$H(\Delta E,\phi) = H(0,\pi-\phi_s).$$

The resulting bucket is

(13) $$\boxed{\begin{aligned}&b(\phi,\phi_s) = \cos\phi + \phi\sin\phi_s + \cos\phi_s - (\pi-\phi_s)\sin\phi_s\\ &+\chi\left[\tfrac{1}{2}\cos(2\phi+\Delta\varphi) + \phi\sin(2\phi_s+\Delta\varphi) - \cos(2\phi_s-\Delta\varphi) - (\pi-\phi_s)\sin(2\phi_s+\Delta\varphi)\right]\end{aligned}}$$

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[1] D.A.Edwards and M.J.Syphers *An Introduction to the Physics of High Energy Accelerators* Wiley, NY 1993, p.38

[2] J.M.Kats *Synchronous Particle and Bucket Dynamics* AGS/AD/89-1, BNL-52171, October 3, 1988

To accommodate the bunch in the bucket, we want to find the two extreme points of the separatrix, ϕ_1 and ϕ_2 ($\phi_2 < \phi_1$). With some manipulation of Eq. (13), a convenient form of the transcendental Equation to be solved to find these points, using as a parameter the bunch length $\Delta\phi$ can be written as

$$\boxed{\sin\psi = A - \frac{\chi}{4\sin(\frac{1}{2}\Delta\phi)}\sin(2\psi + \Delta\varphi)}, \tag{14}$$

with the positions

$$\phi_1 = \psi + \frac{\Delta\phi}{2}, \quad \phi_2 = \phi_1 - \Delta\phi, \quad A = \frac{\frac{1}{2}\Delta\phi}{\sin(\frac{1}{2}\Delta\phi)}\left[\sin\phi_s + \chi\tfrac{1}{2}\Delta\phi\sin(2\phi_s + \Delta\phi)\right] \tag{15}$$

Eq.(4) for the synchronous phase and Eq.(14) for the bucket extreme points can be solved numerically. A very efficient and fast routine for this task is the Newton-Raphson iteration, based on the simple algorithm

$$x_{n+1} = x_n - \frac{f}{f'}, \tag{16}$$

that solves the Equation $f \approx 0$ -f' is the derivative of f- A convenient starting point for the NR iteration, for Eq. (4) is the synchronous phase with no second harmonic

$$\phi_s^{(1)} = \operatorname{ar}\sin\frac{\Delta E_s}{eV_0} \tag{17}$$

(actually a point slightly beyond ϕ_s). A starting point for solving Eq. (14) is

$$\phi_1^{(1)} = \tfrac{1}{2}\Delta\phi + \operatorname{ar}\sin\left(\frac{\frac{1}{2}\Delta\phi}{\sin\frac{1}{2}\Delta\phi}\sin\phi_s\right). \tag{18}$$

Indeed, if $\chi = 0$, the exact solutions of Eqs. (4) and (14) are given by Eqs. (17) and (18), respectively.

Examples of how the synchronous phase is changing by adding a second harmonic to the RF field are shown in Figures 1 and 2. The stable phase is found as the (first positive) zero point of these curves. Figures 3, 4 and 5 show buckets. Parameters are:

$$\frac{eV_0}{\Delta E_s} = 2; \quad \chi = -.75,\ -.5,\ -.25,\ 0,\ .25,\ .5,\ .75.$$

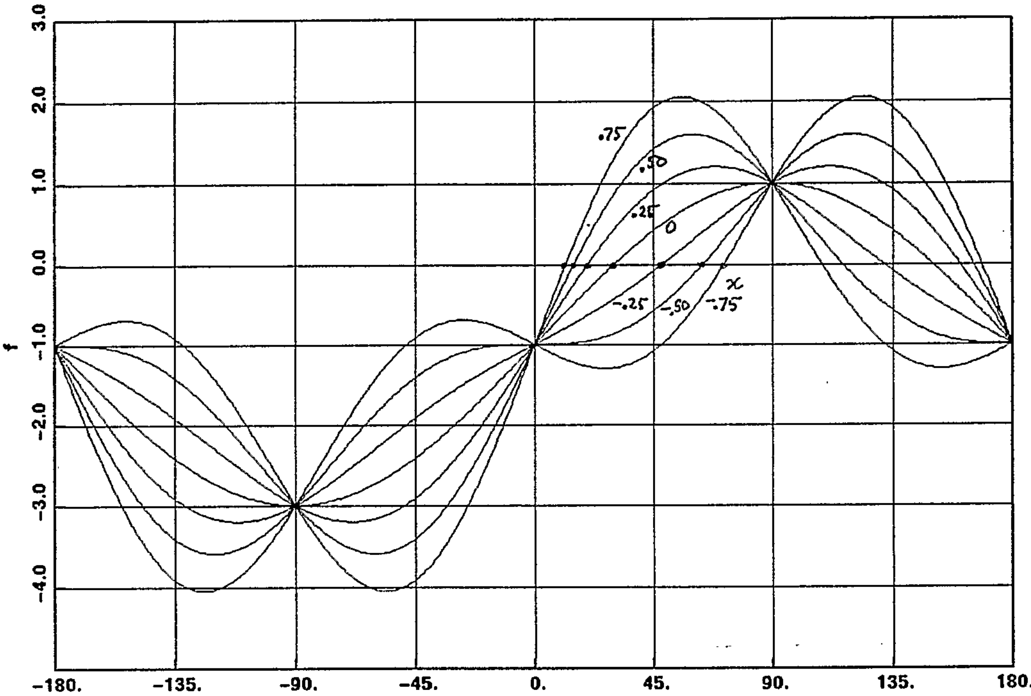

Fig. 1. Synchronous phase ($f = 0$). $\Delta\phi = 0°$.

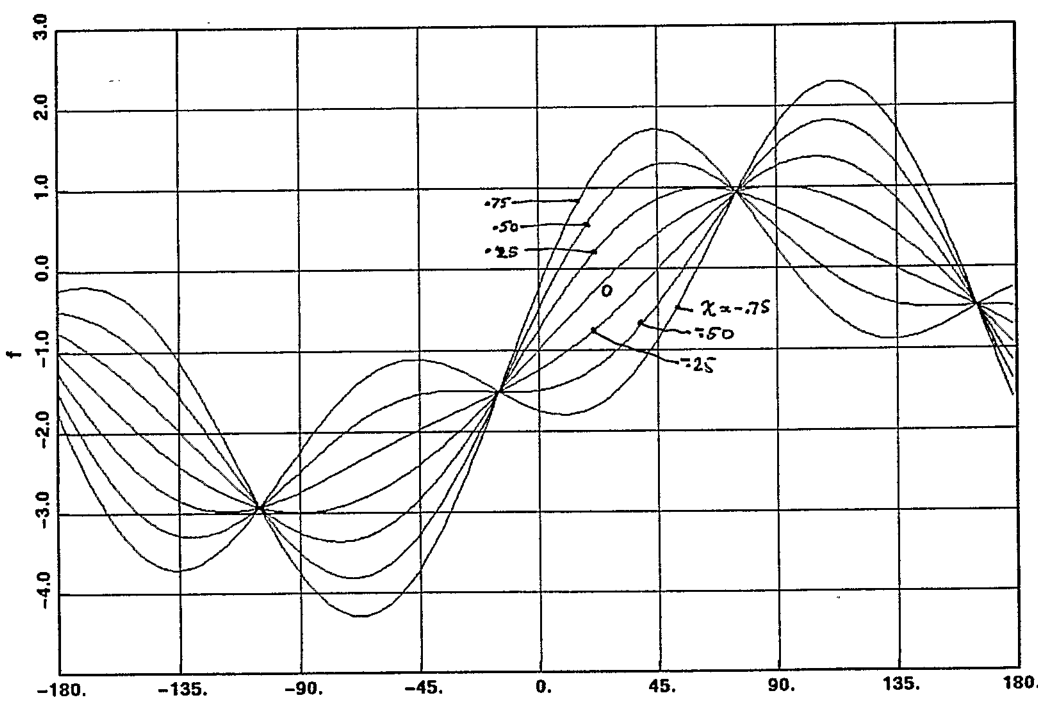

Fig. 2. Synchronous phase ($f = 0$). $\Delta\phi = 30°$.

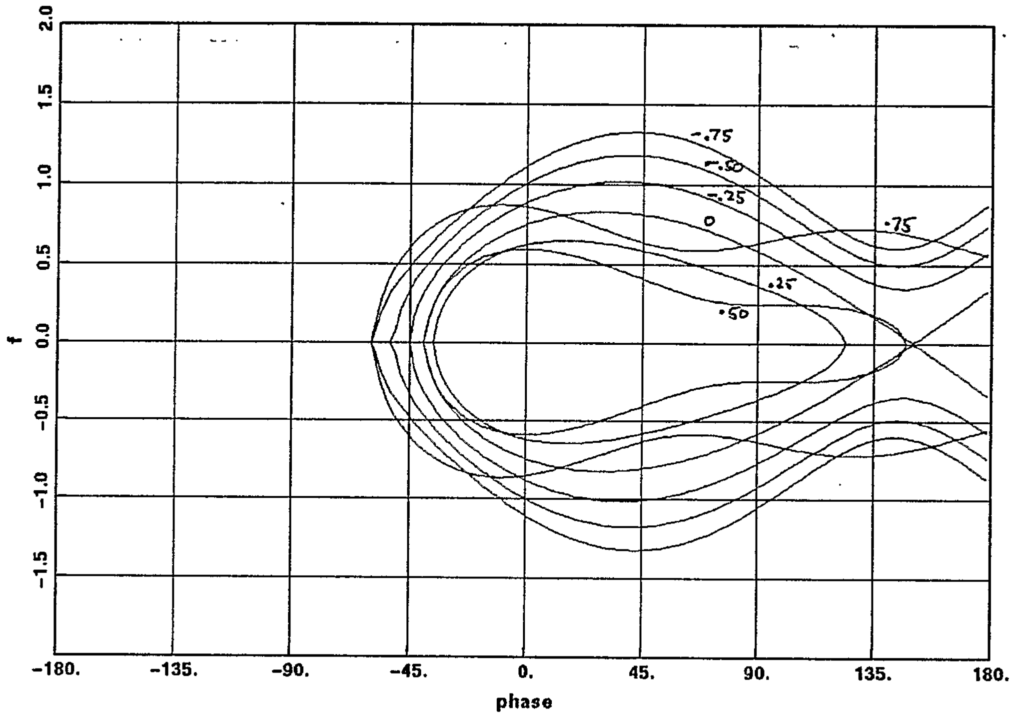

Fig. 3. Buckets: $\Delta\phi = -30°$.

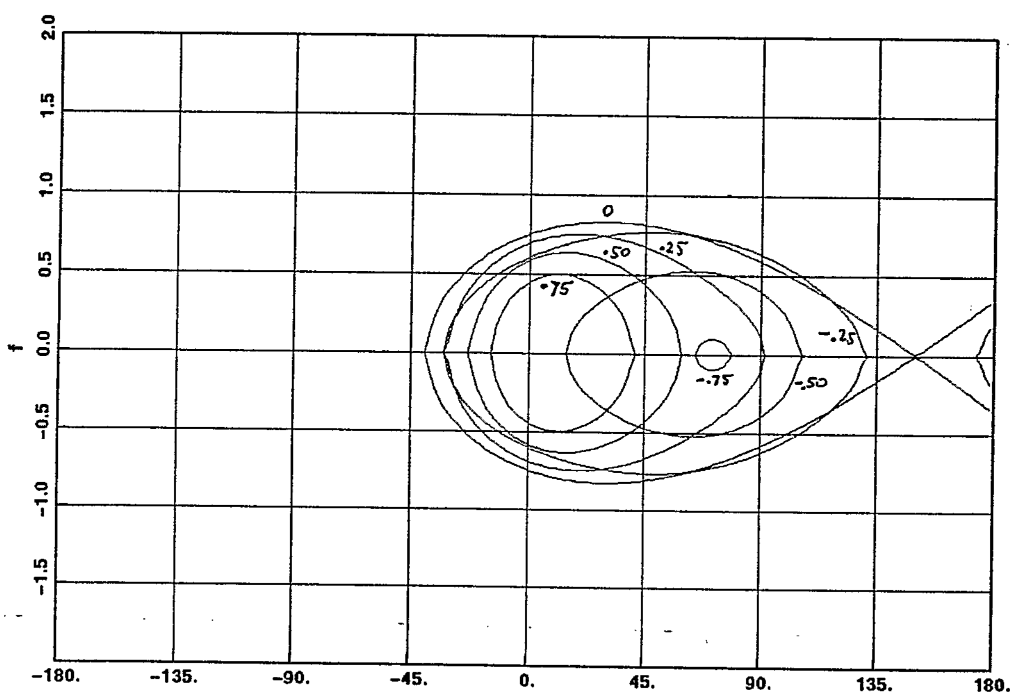

Fig. 4. Buckets: $\Delta\phi = 0°$.

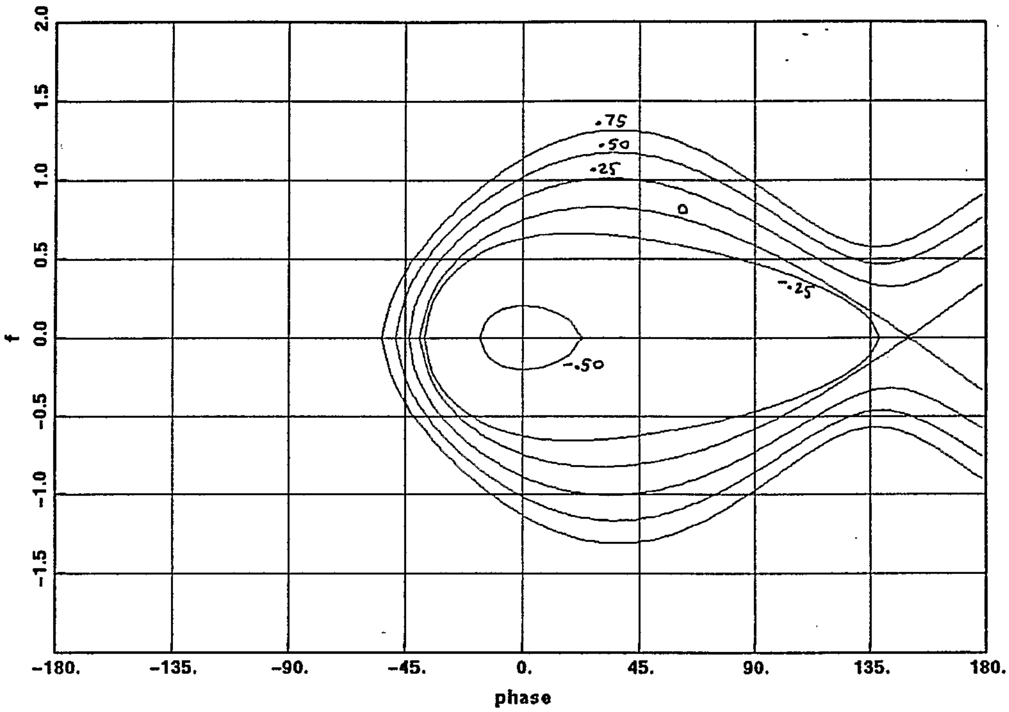

Fig. 5. Buckets: $\Delta\phi = 30°$.